Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational Physics 226 (2007) 1845–1858
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Nyström method for elastic wave scattering by three-dimensional obstacles

Mei Song Tong, Weng Cho Chew *

Center for Computational Electromagnetics and Electromagnetics Laboratory, Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

Received 3 December 2006; received in revised form 7 May 2007; accepted 12 June 2007
Available online 22 June 2007

Abstract

Nyström method is developed to solve for boundary integral equations (BIE's) for elastic wave scattering by three-dimensional obstacles. To generate the matrix equation from a BIE, Nyström method applies a quadrature rule to the integrations of smooth integrands over a discretized element directly and chooses the values of the unknown function at quadrature points as the system's unknowns to be solved. This leads to a simple procedure to form the off-diagonal entries of matrix by simply evaluating the integrands without numerical integrations. For the diagonal or near diagonal entries corresponding to the integrals over a singular or near-singular element where the kernels are singular or near singular, we develop a systematic singularity treatment technique, known as the local correction scheme, based on the linear approximation of elements. The scheme differs from the singularity regularization or subtraction technique used in the boundary element method (BEM). It applies the series expansion of scalar Green's function to the kernels and derives analytical solutions for the strongly singular integrals under the Cauchy principal value like (CPV-like) sense. Since the approach avoids the need for reformulating the BIE for singularity removal in BEM and solves for the Somigliana's equation directly, it is easy to implement and efficient in calculation. Numerical examples are used to demonstrate its robustness.

Keywords: Nyström method; Elastic wave scattering; Boundary integral equation

1. Introduction

The study of the radiation and scattering of elastic wave forms the basis of solution for many physical problems in the realm of elastodynamics such as the dynamic stress concentration, nondestructive testing for materials and earth-structure interaction in an earthquake induced wave environment. These studies usually require solving the wave equations which can be in the form of partial differential equation (PDE) or boundary integral equation (BIE) using numerical procedures. Although the direct solution for

* Corresponding author. Tel.: +1 217 333 7309; fax: +1 217 244 7345.
E-mail address: w-chew@uiuc.edu (W.C. Chew).
URL: http://www.ccem.uiuc.edu (W.C. Chew).

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2007.06.013

the PDE has been a good approach due to its simplicity [1–4], the BIE is more desirably solved in many applications and has received extensive research attention [5–9]. The general BIE for the elastic wave scattering takes the form of the Somigliana's formula and it is usually treated as the starting point for the solution procedure [8,9].

BEM may be the most widely used numerical tool in solving elastic wave BIE's [10–13]. It is a well-established numerical method and is attractive because it reduces the dimensionality of problems if compared with the finite element method (FEM) or the finite difference method (FDM). BEM discretizes the boundary of a source domain and generates an algebraic matrix equation by the collocation method. The key problem in BEM is the accurate numerical integration for the singular integral kernels. For elastic wave scattering problems, the integral kernels in the Somigliana's formula are the Stokes displacement tensor and traction tensor and they are singular over a singular element. The special singularity treatment technique known as the singularity regularization or singularity subtraction has been developed to remove the singularities in the numerical integrations [11]. The technique usually requires reformulating the original Somigliana's equation into a nonsingular or weakly singular form. This obviously increases the complexity in implementation and computational cost in the solving process.

Nyström method was proposed by Nyström [14] and has become an efficient numerical tool in solving integral equations. This method has been introduced in electromagnetics and received much interest [15–17], but has not yet been used in elastodynamics according to our search. The most distinct feature of this method is that it directly evaluates the integrands under a given quadrature rule to generate most matrix entries corresponding to the far interactions between field points and source points after discretizing the integral equation. This is because Nyström method wisely selects the values of the unknown function at quadrature points as the system's unknowns to be solved, and also performs the collocation over those quadrature points. The corresponding entries of coefficient matrix are just the evaluation of the integral kernels times the weights of the quadrature rule. The key problem in the method is also the manipulation of the singular kernels over the singular or near-singular elements. These elements include the self or near interactions between field points and source points and correspond to the diagonal or near diagonal entries in the matrix.

We developed a systematic singularity treatment technique for those self or near interaction terms in the matrix. Such a technique is recognized as the local correction scheme in electromagnetics [15]. Unlike the regularization or subtraction technique in BEM, our local correction is based on the Somigliana's equation without the need of reformulation. We apply the series expansion of scalar Green's functions to the integral kernels and remove the fictitious $\mathcal{O}(1/r^3)$ and $\mathcal{O}(1/r^4)$ singularities in the kernels by cancelling the leading terms, where r is the distance between a field point and a source point. The resultant kernels only have $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r^2)$ singularities. The $\mathcal{O}(1/r)$ singularity can be easily handled numerically by using regularization technique, or equivalently Duffy's method [18]. The strong singularity of $\mathcal{O}(1/r^2)$ type is manipulated under CPV-like sense and we derive the analytical solutions for such singular integrals over a flat triangular element. Due to the concise numerical procedure in generation of coefficient matrix, the Nyström method is simple to implement and efficient in calculation. We present several numerical examples to illustrate the process.

2. Boundary integral equation

Consider the elastic wave scattering problem as shown in Fig. 1, where an elastic obstacle V_2 bounded with S is embedded in an infinite elastic medium V_1. The corresponding parameters of the media are $(\lambda_2, \mu_2, \rho_2)$ for V_2 and $(\lambda_1, \mu_1, \rho_1)$ for V_1 where λ and μ are Lamé constants and ρ is the mass density of the medium. The incident wave is a time–harmonic plane wave propagating along $-x_3$ direction in V_1 and impinging upon the obstacle, yielding the scattered wave in V_1. The BIE for this problem can be derived from the governing PDE of wave propagation in a homogeneous medium [19]

$$(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} + \omega^2\mathbf{u} = -\rho\mathbf{f} \tag{1}$$

where $\mathbf{u}$ is the displacement vector, $\mathbf{f}$ is the body force and ω is the angular frequency. Morse and Feshbach first derived a generalized BIE from the PDE of Holmholtz type by using Huygens' equivalence principle [20]. Pao and Varatharajulu formed the BIE for the elastic wave scattering from (1) in a similar way [19]

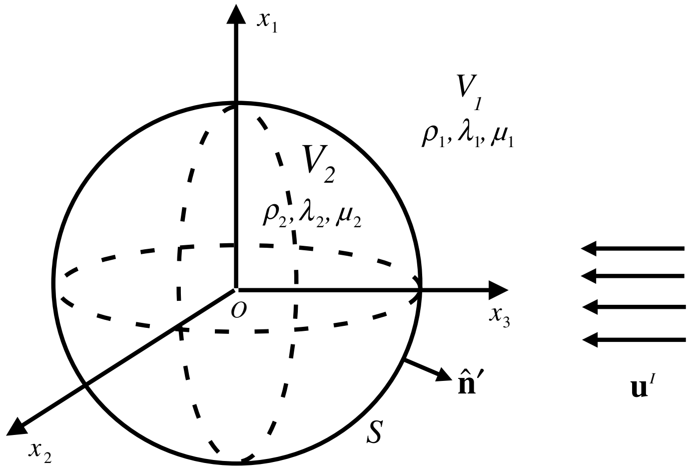

Fig. 1. Elastic wave scattering by an obstacle embedded in an infinite elastic medium.

$$\int_S \{\mathbf{t}(\mathbf{x}') \cdot \overline{\mathbf{G}}(\mathbf{x}, \mathbf{x}') - \mathbf{u}(\mathbf{x}') \cdot [\hat{\mathbf{n}}' \cdot \overline{\overline{\boldsymbol{\Sigma}}}(\mathbf{x}, \mathbf{x}')]\} \, \mathrm{d}S' + \mathbf{u}^I(\mathbf{x}) = \begin{cases} \mathbf{u}(\mathbf{x}), & \mathbf{x} \in V_2 \\ 0, & \mathbf{x} \notin V_2 \end{cases} \tag{2}$$

where $\mathbf{t}$ is the traction vector which can be related to $\mathbf{u}$ by Hooke's law, $\overline{\mathbf{G}}$ is the dyadic Green's function given by

$$\overline{\mathbf{G}} = \frac{1}{\mu}\left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{\kappa_s^2}\right) g_s(\mathbf{x}, \mathbf{x}') - \frac{1}{\gamma}\frac{\nabla\nabla}{\kappa_c^2} g_c(\mathbf{x}, \mathbf{x}') \tag{3}$$

and $\overline{\overline{\boldsymbol{\Sigma}}}(\mathbf{x}, \mathbf{x}')$ is a third-rank Green's tensor with $\Sigma_{\ell mn} = \lambda\delta_{\ell m}\partial_k G_{kn} + \mu(\partial_\ell G_{mn} + \partial_m G_{\ell n})$ in indicial notation. In (3), $g_s = \mathrm{e}^{\mathrm{i}\kappa_s r}/4\pi r$ and $g_c = \mathrm{e}^{\mathrm{i}\kappa_c r}/4\pi r$ are the scalar Green's functions in free space with $r = |\mathbf{x} - \mathbf{x}'|$ being the distance between the field point $\mathbf{x}$ and the source point $\mathbf{x}'$. The subscript s denotes the shear wave and c denotes the compressional wave. The corresponding wave numbers are given by $\kappa_s^2 = \omega^2\rho/\mu$ and $\kappa_c^2 = \omega^2\rho/\gamma$ with $\gamma = \lambda + 2\mu$. In addition, the superscript I in (2) denotes an incident wave, single bar over a vector denotes a dyad, double bars over a vector denote a third-rank tensor and $\bar{\mathbf{I}}$ stands for the identity dyad in (2) and (3).

Eq. (2) is actually the same as the Somigliana's identity [8]

$$\overline{\mathbf{C}}^{\mathrm{T}}(\mathbf{x})\mathbf{u}(\mathbf{x}) = \int_S [\overline{\mathbf{U}}^{\mathrm{T}}(\mathbf{x}, \mathbf{x}')\mathbf{t}(\mathbf{x}') - \overline{\mathbf{T}}^{\mathrm{T}}(\mathbf{x}, \mathbf{x}')\mathbf{u}(\mathbf{x}')] \, \mathrm{d}S' + \mathbf{u}^I(\mathbf{x}) \tag{4}$$

by identifying the Stokes displacement and traction tensors $\overline{\mathbf{U}} = \overline{\mathbf{G}}$ and $\overline{\mathbf{T}} = \hat{\mathbf{n}}' \cdot \overline{\overline{\boldsymbol{\Sigma}}}$. In (4), the superscript T denotes the transpose and $\overline{\mathbf{C}}(\mathbf{x})$ is a tensor which takes the identity dyad for $\mathbf{x}$ in V_1, *0* for $\mathbf{x}$ in V_2, and a real function of the geometry of S in the vicinity of $\mathbf{x}$ for $\mathbf{x}$ on S. If the geometry is smooth at $\mathbf{x}$, then $\overline{\mathbf{C}}(\mathbf{x}) = \bar{\mathbf{I}}/2$. The Stokes tensors can be expressed in an indicial notation [8]

$$\begin{aligned} U_{ij} &= (\overline{\mathbf{G}})_{ij} = \frac{1}{4\pi\rho\omega^2 r^3}\left\{\delta_{ij}[(\kappa_s r)^2 \mathrm{e}^{\mathrm{i}\kappa_s r} + D] + C\partial_i r \partial_j r\right\} \\ T_{ij} &= (\hat{\mathbf{n}}' \cdot \overline{\overline{\boldsymbol{\Sigma}}})_{ij} = \frac{1}{4\pi\rho\omega^2 r^4}\left\{\lambda \mathrm{e}^{\mathrm{i}\kappa_c r}(\kappa_c r)^2(\mathrm{i}\kappa_c r - 1)n_i\partial_j r + \mu \mathrm{e}^{\mathrm{i}\kappa_s r}(\kappa_s r)^2(\mathrm{i}\kappa_s r - 1)\left(\delta_{ij}\frac{\partial r}{\partial n} + n_i\partial_j r\right)\right. \\ &\quad \left. + 2\mu\left[C\left(\delta_{ij}\frac{\partial r}{\partial n} + n_i\partial_j r + n_j\partial_i r\right) + F\partial_i r\partial_j r\frac{\partial r}{\partial n}\right]\right\} \end{aligned} \tag{5}$$

where δ_{ij} is the Kronecker delta, $\partial_i r = \partial r/\partial x_i$, $\partial_j r = \partial r/\partial x_j$, and

$$
\begin{aligned}
C &= \Omega_s \mathrm{e}^{\mathrm{i}\kappa_s r} - \Omega_c \mathrm{e}^{\mathrm{i}\kappa_c r} \\
D &= (\mathrm{i}\kappa_s r - 1)\mathrm{e}^{\mathrm{i}\kappa_s r} - (\mathrm{i}\kappa_c r - 1)\mathrm{e}^{\mathrm{i}\kappa_c r} \\
F &= H_c \mathrm{e}^{\mathrm{i}\kappa_c r} - H_s \mathrm{e}^{\mathrm{i}\kappa_s r} \\
\Omega_s &= 3 - 3\mathrm{i}\kappa_s r - \kappa_s^2 r^2 \\
\Omega_c &= 3 - 3\mathrm{i}\kappa_c r - \kappa_c^2 r^2 \\
H_s &= 15 - 15\mathrm{i}\kappa_s r - 6\kappa_s^2 r^2 + \mathrm{i}\kappa_s^3 r^3 \\
H_c &= 15 - 15\mathrm{i}\kappa_c r - 6\kappa_c^2 r^2 + \mathrm{i}\kappa_c^3 r^3
\end{aligned} \tag{6}
$$

The Stokes tensors are smooth for far-interaction terms and the related integrals can be performed directly using numerical integrations. However, they possess fictitious $\mathcal{O}(1/r^3)$ and $\mathcal{O}(1/r^4)$ singularities on singular elements and the numerical integrations cannot be applied directly. If we expand the scalar Green's function g_s and g_c in (3) into a series form

$$
g(\mathbf{x}, \mathbf{x}') = \frac{\mathrm{e}^{\mathrm{i}kr}}{r} = \sum_{m=0}^{\infty} \frac{(\mathrm{i}\kappa)^m r^{m-1}}{m!} \approx \sum_{m=0}^{M} \frac{(\mathrm{i}\kappa)^m r^{m-1}}{m!} \tag{7}
$$

then the fictitious singularities can be removed by cancelling the leading terms in $\overline{\mathbf{G}}$ and the Stokes tensors become

$$
\begin{aligned}
U_{ij} &\approx \frac{1}{4\pi\rho\omega^2}\left\{\left[\frac{2}{r}\left(\kappa_s^2 - \kappa_c^2\right) - \mathrm{i}\left(\kappa_s^3 - \kappa_c^3\right)\right]\partial_i r \partial_j r + \frac{\delta_{ij}}{r}\left(\kappa_c^2 - \kappa_s^2 + \kappa_s^2 \mathrm{e}^{\mathrm{i}\kappa_s r}\right)\right. \\
&\quad \left. + \sum_{m=2}^{M} \frac{\mathrm{i}^m r^{m-3}}{m!}\left[\left(\kappa_s^m \Omega_s - \kappa_c^m \Omega_c\right)\partial_i r \partial_j r + \delta_{ij}\kappa_s^m(\mathrm{i}\kappa_s r - 1) - \delta_{ij}\kappa_c^m(\mathrm{i}\kappa_c r - 1)\right]\right\} \\
T_{ij} &\approx \frac{1}{4\pi\rho\omega^2}\left[\lambda\kappa_c^2 n_i \partial_j r A + \mu\kappa_s^2\left(\delta_{ij}\frac{\partial r}{\partial n} + n_j \partial_i r\right)B + 2\mu\left(\delta_{ij}\frac{\partial r}{\partial n}P + n_i \partial_j r P + n_j \partial_i r P + \frac{\partial r}{\partial n}\partial_i r \partial_j r Q\right)\right]
\end{aligned} \tag{8}
$$

where

$$
\begin{aligned}
A &= \frac{(\mathrm{i}\kappa_c r - 1)}{r^2}\mathrm{e}^{\mathrm{i}\kappa_c r} = -\frac{1}{r^2} - \kappa_c^2 + \sum_{m=2}^{M}\frac{(\mathrm{i}\kappa_c)^m}{m!}(\mathrm{i}\kappa_c r - 1)r^{m-2} \\
B &= \frac{(\mathrm{i}\kappa_{sr-1)} r^2}{}\mathrm{e}^{\mathrm{i}\kappa_s r} = -\frac{1}{r^2} - \kappa_s^2 + \sum_{m=2}^{M}\frac{(\mathrm{i}\kappa_s)^m}{m!}(\mathrm{i}\kappa_s r - 1)r^{m-2} \\
P &= \frac{1}{r^4}\left(\Omega_s \mathrm{e}^{\mathrm{i}\kappa_s r} - \Omega_c \mathrm{e}^{\mathrm{i}\kappa_c r}\right) = \frac{1}{2r^2}\left(\kappa_s^2 - \kappa_c^2\right) + \frac{\mathrm{i}r}{6}\left(\kappa_s^5 - \kappa_c^5\right) + \sum_{m=2}^{M}\frac{\mathrm{i}^m}{m!}\left(\kappa_s^m \Omega_s - \kappa_c^m \Omega_c\right)r^{m-4} \\
Q &= \frac{1}{r^4}\left(H_c \mathrm{e}^{\mathrm{i}\kappa_c r} - H_s \mathrm{e}^{\mathrm{i}\kappa_s r}\right) = \frac{3}{2r^2}\left(\kappa_c^2 - \kappa_s^2\right) - \frac{1}{2}\left(\kappa_c^4 - \kappa_s^4\right) + \frac{\mathrm{i}}{2}\left(\kappa_c^5 - \kappa_s^5\right)r + \frac{1}{6}\left(\kappa_c^6 - \kappa_s^6\right)r^2 \\
&\quad + \sum_{m=4}^{M}\frac{\mathrm{i}^m}{m!}\left(\kappa_s^m \Omega_s - \kappa_c^m \Omega_c\right)r^{m-4}
\end{aligned} \tag{9}
$$

In the new form of the Stokes tensors, the $\mathcal{O}(1/r)$ singularity in U_{ij} can be handled using the regularization technique in BEM, or equivalently the Duffy's method [18] in electromagnetics. The $\mathcal{O}(1/r^2)$ singularity in T_{ij} is handled using the subtraction technique in BEM, but we derive closed-form formulas in a CPV-like sense. Since the new form of the Stokes tensors is only applied to the singular or near-singular elements where r is small, the series converges fast and the number of truncated terms M is small.

If we incorporate boundary conditions which are the continuity of displacement and traction vectors $\mathbf{u}$ and $\mathbf{t}$ on S, the BIE's in indicial notation can be written as

$$\begin{aligned}
&\int_S \left[T_{ij}^{(1)}(\mathbf{x},\mathbf{x}')u_j(\mathbf{x}') - U_{ij}^{(1)}(\mathbf{x},\mathbf{x}')t_j(\mathbf{x}')\right] \mathrm{d}S' + \frac{1}{2}\delta_{ij}u_i(\mathbf{x}) = u_i^{\mathrm{I}}(\mathbf{x}), \quad \mathbf{x} \in S \\
&\int_S \left[U_{ij}^{(2)}(\mathbf{x},\mathbf{x}')t_j(\mathbf{x}') - T_{ij}^{(2)}(\mathbf{x},\mathbf{x}')u_j(\mathbf{x}')\right] \mathrm{d}S' + \frac{1}{2}\delta_{ij}u_i(\mathbf{x}) = 0, \quad \mathbf{x} \in S
\end{aligned} \tag{10}$$

on which the numerical procedure is based. The superscripts on the Stokes tensors denote the media and we choose the field point $\mathbf{x}$ on a smooth surface leading to the coefficient of $\delta_{ij}/2$ in front of $u_i(\mathbf{x})$. If the obstacle is a traction-free cavity or a fixed rigid inclusion, the above BIE's can be simplified by enforcing $\mathbf{t} = 0$ or $\mathbf{u} = 0$ on S.

3. Nyström method

In BIE, when we transform the integral equation into a matrix equation after discretizing the geometry, we usually express the unknown function into interpolation functions with unknown coefficients. Replacing the unknown function with such an expression leads to the integration of the kernels with the interpolation functions for each matrix entry. The interpolation functions are chosen as shape functions of an element in an isoparametric scheme and the unknown coefficients are the system's unknowns to be solved. Nyström method uses a different strategy to generate the matrix entries. If the integral kernels are smooth over an element, Nyström method replaces the integral with a summation under a quadrature rule, i.e.

$$\int_{\Delta S} f(\mathbf{x}')\,\mathrm{d}S' = \sum_{j=1}^{P} w_j f(\mathbf{x}'_j) \tag{11}$$

where $f(\mathbf{x})$ is a general smooth function, P is the number of quadrature points and w_j is the jth weight over the surface element ΔS which can be a triangle, square, or circle, etc. We usually use the non-product quadrature rule instead of product or repeated quadrature rule for a surface integral because the non-product rule has less quadrature points. The typical non-product rules are the Gauss–Legendre rule with one, three or six quadrature points over a flat triangle [21]. Nyström method wisely selects the values of unknown function at quadrature points as the unknowns of the matrix equation and the matrix entries are just the direct evaluation of integral kernels times the weights of the quadrature rule. Consider a generalized 3D BIE

$$\int_S F(\mathbf{x},\mathbf{x}')u(\mathbf{x}')\,\mathrm{d}S' = -\phi(\mathbf{x}) \quad \mathbf{x} \in S \tag{12}$$

where $F(\mathbf{x},\mathbf{x}')$ is the integral kernel related to the scalar Green's functions, $u(\mathbf{x}')$ is the unknown function, say the component of displacement vector, and $\phi(\mathbf{x})$ represents the excitation (incident wave). After discretizing the surface S into N patches and applying the quadrature rule in (11), the BIE becomes

$$\sum_{i=1}^{N}\sum_{j=1}^{P_i} w_{ij}F(\mathbf{x},\mathbf{x}'_{ij})u(\mathbf{x}'_{ij}) = -\phi(\mathbf{x}) \tag{13}$$

where P_i is the number of quadrature points at the ith element and w_{ij} represents the weight of the quadrature rule on the jth point of the ith element. Performing the collocation procedure on these quadrature points yields the following matrix equation

$$\sum_{i=1}^{N}\sum_{j=1}^{P_i} w_{ij}F(\mathbf{x}_{mn},\mathbf{x}'_{ij})u(\mathbf{x}'_{ij}) = -\phi(\mathbf{x}_{mn}) \tag{14}$$

where $m = 1,2,\ldots,N$ and $n = 1,2,\ldots,P_m$. We can see that the whole process is very simple because it avoids the need of interpolations and numerical integrations. Since the integrands are usually smooth for far-interaction elements, the matrix entries corresponding to these elements can be created in such a way. This will greatly facilitate the implementation and save on matrix-filling time. For singular or near-singular elements, the integrands are singular or badly behaved and the quadrature rule in (11) cannot be applied directly. A special treatment called local correction is needed to generate those diagonal or near diagonal entries. The local correction is actually a singularity manipulation technique and very similar to the procedure in BEM. The difference is that we interpolate the unknown function over a singular element based on the values of unknown function at quadrature points and we have developed analytical solutions for the resultant strongly singular integrals under a CPV-like sense.

4. Strongly singular integrals

Since the weakly singular integrals with $\mathcal{O}(1/r)$ singularity can be handled by regularization technique or Duffy's method, we will not discuss them here. We focus on those strongly singular integrals with $\mathcal{O}(1/r^2)$ singularity. These integrals are handled using the singularity subtraction technique in BIE, but we derive the closed-form expressions here. If we combine the interpolation function with the integral kernel T_{ij} in (8), the strongly singular integrals are the following terms in T_{ij}

$$f_1 = \frac{1}{r^2}\frac{\partial r}{\partial x_i}; \quad f_2 = \frac{1}{r^2}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j}\frac{\partial r}{\partial n} \tag{15}$$

The resultant integrals take the following forms in a global coordinate system (x_1, x_2, x_3):

$$\begin{aligned}
&I_1 = \int_{\Delta S}\frac{(x_1-x_1')}{r^3}\,dS'; \quad I_2 = \int_{\Delta S}\frac{(x_2-x_2')}{r^3}\,dS'; \quad I_3 = \int_{\Delta S}\frac{(x_3-x_3')}{r^3}\,dS';\\
&I_4 = \int_{\Delta S}\frac{(x_1-x_1')^3}{r^5}\,dS'; \quad I_5 = \int_{\Delta S}\frac{(x_2-x_2')^3}{r^5}\,dS'; \quad I_6 = \int_{\Delta S}\frac{(x_3-x_3')^3}{r^5}\,dS';\\
&I_7 = \int_{\Delta S}\frac{(x_1-x_1')^2(x_2-x_2')}{r^5}\,dS'; \quad I_8 = \int_{\Delta S}\frac{(x_1-x_1')(x_2-x_2')^2}{r^5}\,dS';\\
&I_9 = \int_{\Delta S}\frac{(x_1-x_1')^2(x_3-x_3')}{r^5}\,dS'; \quad I_{10} = \int_{\Delta S}\frac{(x_1-x_1')^2(x_3-x_3')}{r^5}\,dS';\\
&I_{11} = \int_{\Delta S}\frac{(x_2-x_2')^2(x_3-x_3')}{r^5}\,dS'; \quad I_{12} = \int_{\Delta S}\frac{(x_2-x_2')(x_3-x_3')^2}{r^5}\,dS';\\
&I_{13} = \int_{\Delta S}\frac{(x_1-x_1')(x_2-x_2')(x_3-x_3')}{r^5}\,dS'
\end{aligned} \tag{16}$$

We create a local coordinate system (u, v, w) over the singular element as shown in Fig. 2. The transformation relation between the global and local coordinate system is

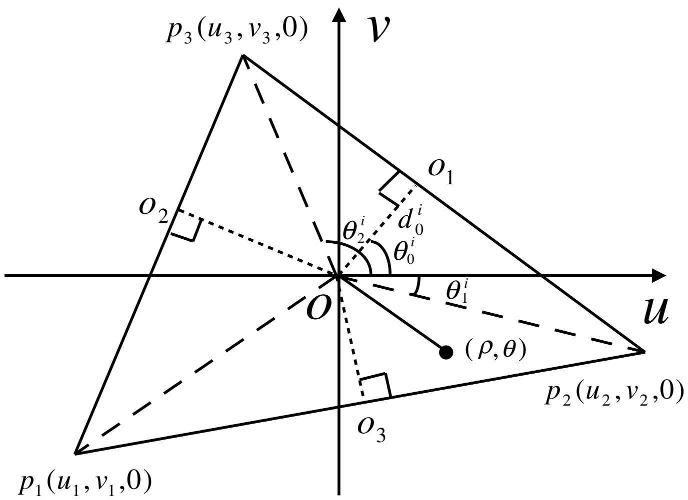

Fig. 2. Local coordinate system over a singular element. The field point is initially located at $(0, 0, w)$ and approaches the element with $w \to 0$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} \tag{17}$$

where $(\ell_1, m_1, n_1), (\ell_2, m_2, n_2)$ and (ℓ_3, m_3, n_3) are the direction cosines of three local coordinate axes in the global coordinate system and (u_0, v_0, w_0) is the coordinates of the local coordinate system's origin in the global coordinate system. In this local coordinate system, those strongly singular integrals can be expressed as

$$\begin{aligned} I_k &= a_k^{(1)} I_u^{(3)} + a_k^{(2)} I_v^{(3)} \quad \text{for } k = 1, 2, 3; \\ I_k &= a_k^{(3)} I_u^{(5)} + a_k^{(4)} I_{uv}^{(5)} + a_k^{(5)} I_{vu}^{(5)} + a_k^{(6)} I_v^{(5)} \quad \text{for } k = 4, \ldots, 13 \end{aligned} \tag{18}$$

where

$$\begin{aligned} I_u^{(3)} &= \int_{\Delta S} \frac{u'}{r^3} \, \mathrm{d}S'; \quad I_v^{(3)} = \int_{\Delta S} \frac{v'}{r^3} \, \mathrm{d}S'; \quad I_u^{(5)} = \int_{\Delta S} \frac{u'^3}{r^5} \, \mathrm{d}S'; \quad I_{uv}^{(5)} = \int_{\Delta S} \frac{u'^2 v'}{r^5} \, \mathrm{d}S'; \\ I_{vu}^{(5)} &= \int_{\Delta S} \frac{u' v'^2}{r^5} \, \mathrm{d}S'; \quad I_v^{(5)} = \int_{\Delta S} \frac{v'^3}{r^5} \, \mathrm{d}S' \end{aligned} \tag{19}$$

and $a_k^{(i)} (i = 1, \ldots, 6)$ are the constants related to the direction cosines. The integrals in (19) are strongly singular and the accurate evaluation for them is the key part of our approach. We evaluate these integrals under a CPV-like sense, i.e., we assume that the field point is initially off the singular element with a vertical distance of w from the singular element plane. After deriving the analytical solution for those integrals with $w \neq 0$ under the polar coordinate system as shown in Fig. 2, we take the limit of $w \to 0$ to find the exact solutions for the integrals in (19). The final results can be found in the Appendix.

5. Numerical results

Several numerical examples are presented to demonstrate the validity of the Nyström method. The one-point quadrature rule which corresponds to a constant approximation for the unknown function over an element is used in all cases. The higher-order quadrature rules corresponding to a higher-order interpola-

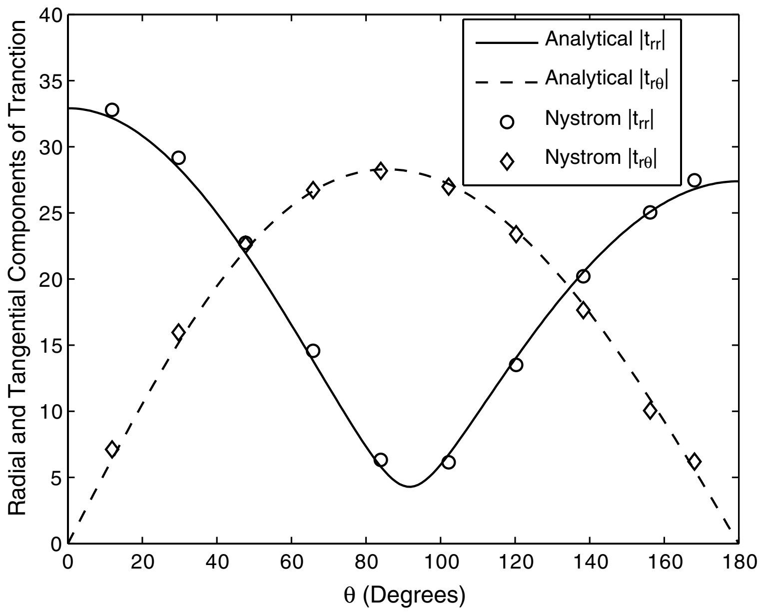

Fig. 3. Total radial and tangential traction along the elevated cut at the surface of a rigid sphere, $k_c a = 0.125$.

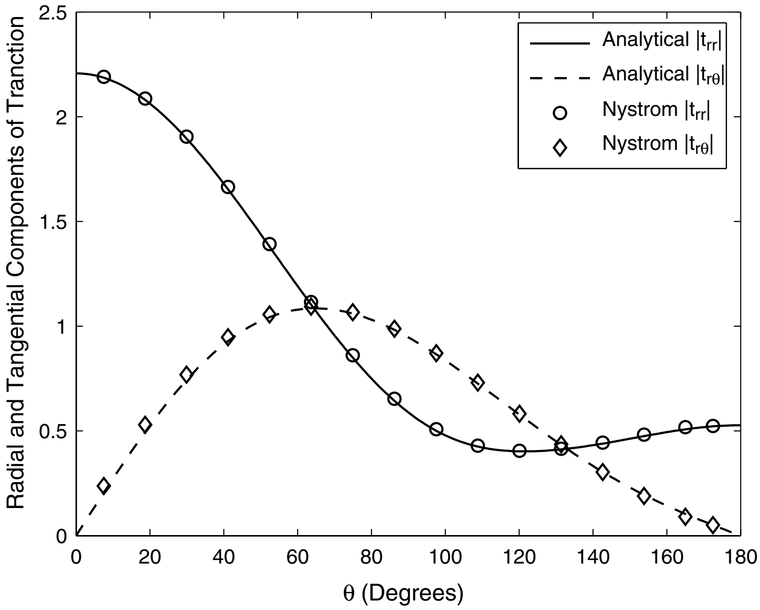

Fig. 4. Total radial and tangential traction along the elevated cut at the surface of a rigid sphere, $k_c a = 0.913$.

tion for the unknown function can be used in a straightforward way. This is because the strongest singularity comes from the constant term of the interpolation function. The higher-order terms of interpolation functions will weaken the degree of singularity when exerting on the kernels and the resultant integrals can all be handled numerically. The tangential component in the following denotes the θ component in all cases.

We first consider the scattering by a fixed rigid sphere with a radius of $a = 1.0$. The surrounding medium has Poisson's ratio $v = 0.25$ and mass density $\rho = 1.0$. The incident wave is a time–harmonic plane dilatational wave with a unit circular frequency ($\omega = 1.0$) and normalized wave number of $\kappa_c a = 0.125$, 0.913 and π, respectively. We use 360, 960 and 960 triangle elements, respectively, in the discretization of the geometry

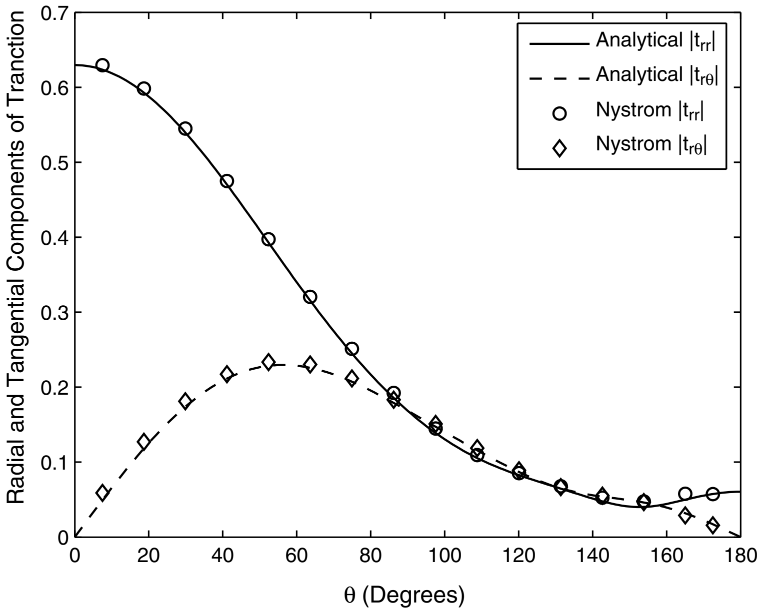

Fig. 5. Total radial and tangential traction along the elevated cut at the surface of a rigid sphere, $k_c a = \pi$.

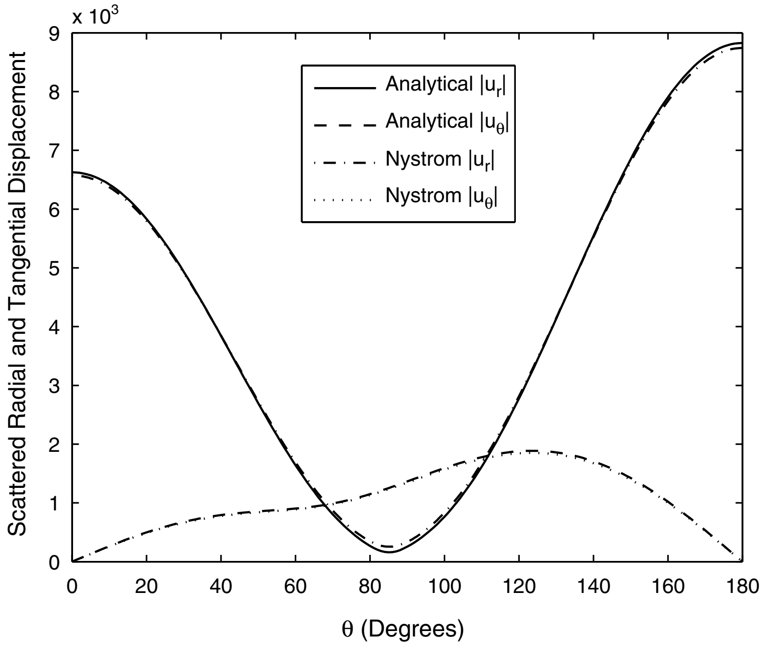

Fig. 6. Scattered radial and tangential displacement of a spherical cavity at $r = 5a$, $k_c a = 0.09$.

for these wave numbers. The meshes are over fine for the $\kappa_c a = 0.125$ and 0.913 cases (small and middle sizes) because we want to show more points on the curves. Figs. 3–5 show the total radial and tangential components of traction along the principal cut ($\phi = 0°$ and $\theta = 0°–180°$) at the surface. It can be seen that the solutions agree with the analytical solutions very well. The analytical solutions can be found in [22–24].

We next consider the scattering by a traction-free spherical cavity surrounded by an infinite elastic medium. The cavity also has a radius of $a = 1.0$ and the surrounding medium is characterized by Poisson's ratio $v = 1/3$, Young's modulus $E = 2/3$ and mass density $\rho = 1.0$. The incident wave is the same as before but the normalized

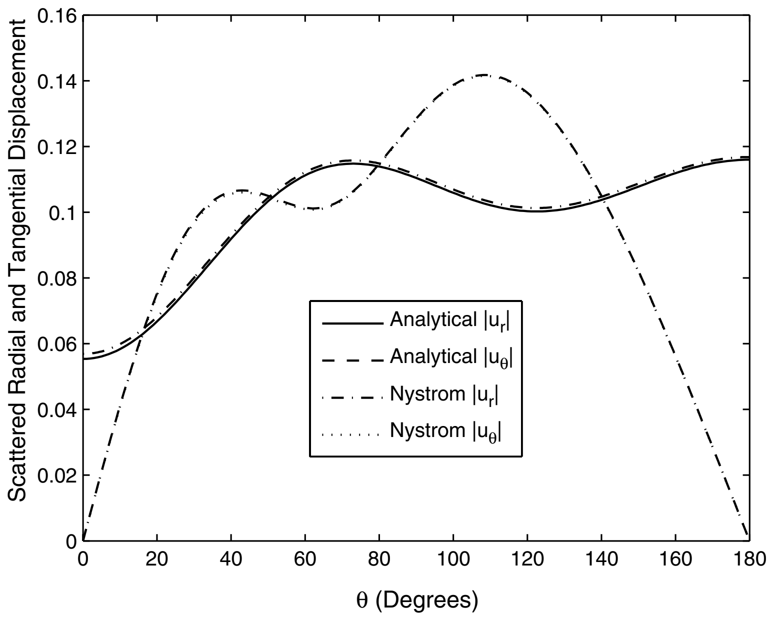

Fig. 7. Scattered radial and tangential displacement of a spherical cavity at $r = 5a$, $k_c a = 0.913$.

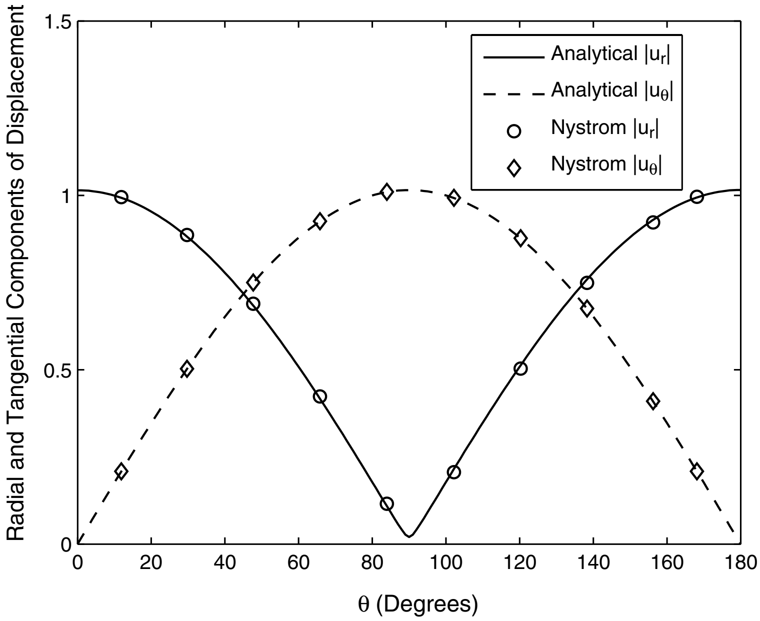

Fig. 8. Total radial traction and displacement along the elevated cut at the surface of an elastic sphere, $k_c a = 0.125$.

wave numbers are chosen as $\kappa_c a = 0.09$ and 0.913 corresponding to 360 and 960 triangle meshes in the discretization, respectively. Figs. 6 and 7 show the scattered radial and tangential displacements along the principal cut at $r = 5a$. The solutions are also compared with the analytical solutions and they are very close to each other.

We finally consider the generalized case with both the surrounding medium and obstacle being elastic. We select $\lambda_1 = 0.53486$, $\mu_1 = 0.23077$ and $\rho_1 = 1.0$ for the surrounding medium, and $\lambda_2 = 0.23716$, $\mu_2 = 0.52641$ and $\rho_2 = 1.9852$ for the elastic spherical obstacle with a unit radius. The incident wave is also the same as before but the normalized wave numbers are chosen as $\kappa_c a = 0.125$, 2.0 and 5.0, respectively. Figs. 8–10 plot the total radial components of displacement and traction at the surface along the principal cut. These results,

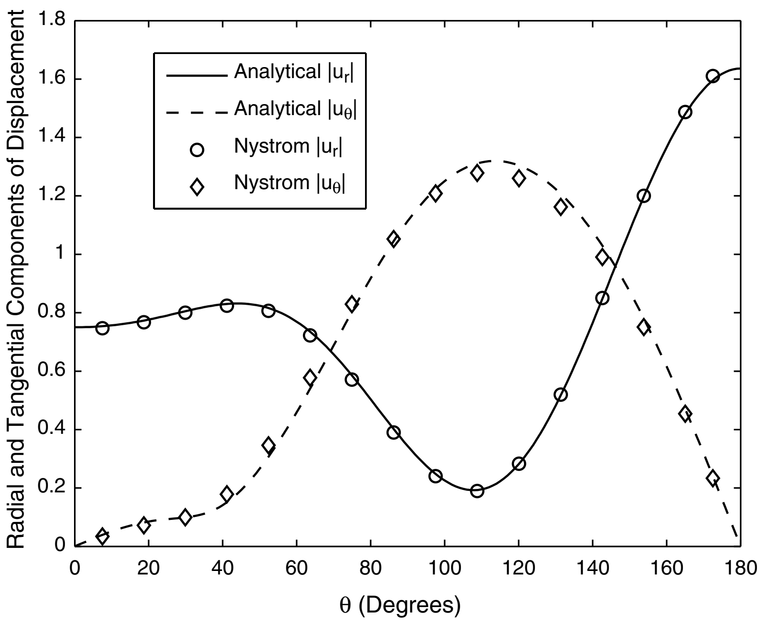

Fig. 9. Total radial traction and displacement along the elevated cut at the surface of an elastic sphere, $k_c a = 2.0$.

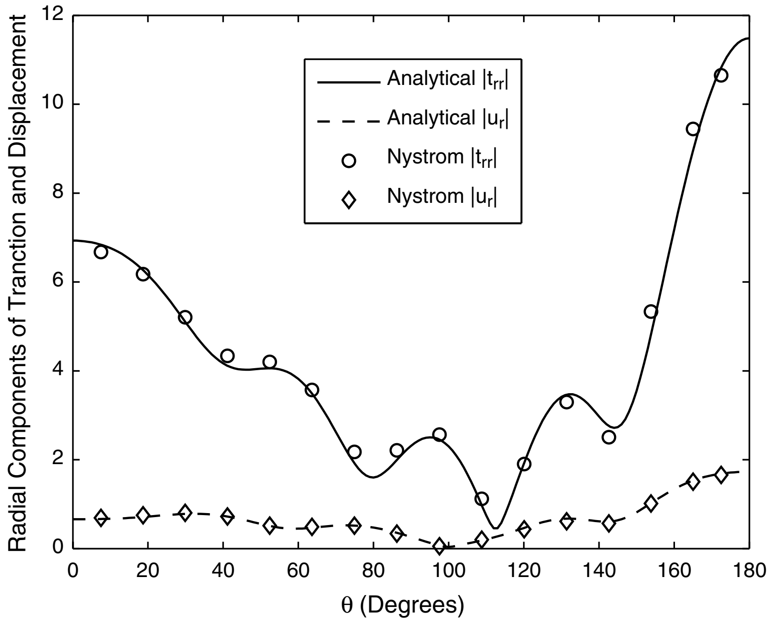

Fig. 10. Total radial traction and displacement along the elevated cut at the surface of an elastic sphere, $k_c a = 5.0$.

obtained using 360, 960 and 960 elements, respectively, are again in excellent agreement with the analytical solutions. Note that $\kappa_c a = 4.4934$ is a wave number close to the fictitious eigenfrequency in this case and it is hard to obtain a good result using the traditional BEM [25]. To get rid of the resonance problem, one usually needs to reformulate the integral equation and a higher cost is required in the solution process. However, like the method of fundamental solution (MFS) [26], our Nyström method solution does not sharply worsen near the fictitious eigenfrequency when solving the original equation. The results using 960 elements are still good enough as shown in Fig. 11. The insensitiveness of eigenfrequency may allow the method to be a good

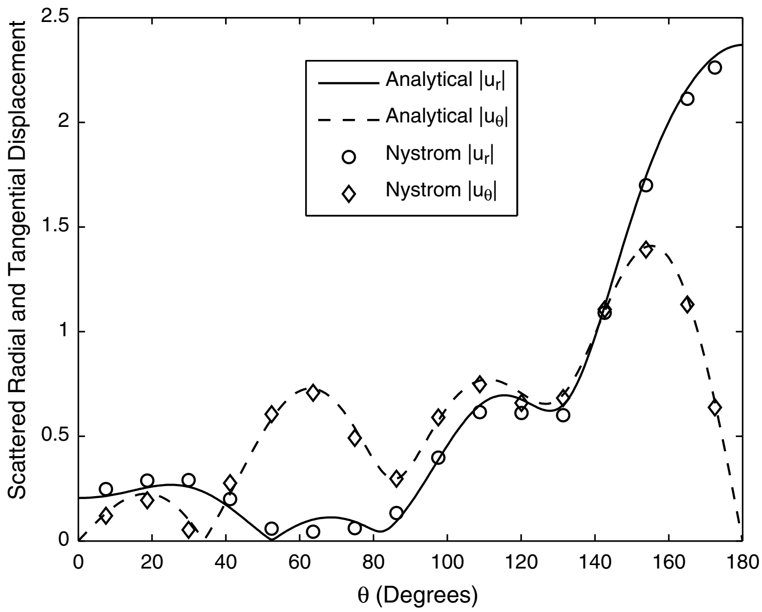

Fig. 11. Scattered displacement of an elastic sphere along the principal cut at the surface, $k_c a = 4.4934$.

choice in seeking numerical solutions. Therefore, this method does not remove the internal resonance problem, but make the solution robust close to the resonance frequency.

6. Conclusion

Nyström method was originally designed for solving integral equations with smooth kernels. The singular feature of the Green's function in BIE's has prevented it from being exploited for a long time. It has not been used as a numerical tool until the robust local correction schemes are developed. In this work, we have developed a local correction scheme for solving elastic wave scattering problems. This scheme differs from the singularity treatment technique in BEM and it is simpler to implement. We apply the series expression of the scalar Green's functions to the Stokes tensors and cancel the fictitious higher-order singular terms. The remaining $\mathcal{O}(1/r)$ singularity can be easily handled numerically, but the $\mathcal{O}(1/r^2)$ singularity are performed analytically under a CPV-like sense. This approach avoids the need to reformulate the Somigliana's equation when kernels are singular and also avoids the need for numerical integrations when kernels are smooth. The simplified singularity treatment technique, together with the simpler process of generating the nonsingular entries of matrix, constitutes the main attractiveness of this method. The method is only implemented in linear elements (flat triangles) currently. We will consider the higher-order approximation of elements for curvilinear geometry in the future.

Acknowledgment

This work was supported by CERL, US Army under Grant W9132T-06-2-0006.

Appendix. Analytical solutions for the strongly singular integrals in (19)

In the polar coordinate system as shown in Fig. 2, we divide the triangle element into three sub-triangles ΔS_i $(i = 1, 2, 3)$ by connecting the origin to the three vertices of the triangle element. Each sub-triangle is specified by four parameters d_0^i, θ_0^i, θ_1^i and θ_2^i. Here, d_0^i is the vertical distance $\overline{OO_i}$ between the origin and the ith side of the triangle element, θ_0^i is the angle of $\overline{OO_i}$, and θ_k^i $(k = 1, 2)$ are the angles of connecting lines from the origin to the two end points of the ith side, respectively. With the aid of formulas in [27], those integrals can be derived as follows:

$$\begin{aligned}
I_u^{(3)} &= \int_{\Delta S} \frac{u'}{r^3}\,\mathrm{d}S' = \sum_{i=1}^{3}\left[\left(\sin\theta_2^i - \sin\theta_1^i\right)\ln\left(d_0^i\right) - I_a\cos\theta_0^i + I_b\sin\theta_0^i\right]\\
I_v^{(3)} &= \int_{\Delta S} \frac{v'}{r^3}\,\mathrm{d}S' = \sum_{i=1}^{3}\left[\left(\cos\theta_1^i - \cos\theta_2^i\right)\ln\left(d_0^i\right) - I_b\cos\theta_0^i - I_a\sin\theta_0^i\right]\\
I_u^{(5)} &= \int_{\Delta S} \frac{u'^3}{r^5}\,\mathrm{d}S' = \sum_{i=1}^{3}\left[\left(\sin\theta_2^i - \sin\theta_1^i + t_1\right)\ln\left(d_0^i\right)\right.\\
&\quad \left. - I_A\cos^3\theta_0^i + 1.5\sin 2\theta_0^i\left(I_B\cos\theta_0^i - I_C\sin\theta_0^i\right) + I_D\sin^3\theta_0^i\right]\\
I_{uv}^{(5)} &= \int_{\Delta S} \frac{u'^2 v'}{r^5}\,\mathrm{d}S' = \sum_{i=1}^{3}\left[t_2\ln\left(d_0^i\right) - I_A\sin\theta_0^i\cos^2\theta_0^i - I_D\cos\theta_0^i\sin^2\theta_0^i\right.\\
&\quad \left. + I_B\cos\theta_0^i\left(3\sin^2\theta_0^i - 1\right) + I_C\sin\theta_0^i\left(3\cos^2\theta_0^i - 1\right)\right]\\
I_{vu}^{(5)} &= \int_{\Delta S} \frac{u' v'^2}{r^5}\,\mathrm{d}S' = \sum_{i=1}^{3}\left[I_D\sin\theta_0^i\cos^2\theta_0^i - I_A\cos\theta_0^i\sin^2\theta_0^i - t_1\ln\left(d_0^i\right)\right.\\
&\quad \left. - I_B\sin\theta_0^i\left(3\cos^2\theta_0^i - 1\right) + I_C\cos\theta_0^i\left(3\sin^2\theta_0^i - 1\right)\right]\\
I_v^{(5)} &= \int_{\Delta S} \frac{v'^3}{r^5}\,\mathrm{d}S' = \sum_{i=1}^{3}\left[\left(\cos\theta_1^i - \cos\theta_2^i - t_2\right)\ln\left(d_0^i\right) - I_A\sin^3\theta_0^i\right.\\
&\quad \left. - 1.5\sin 2\theta_0^i\left(I_B\sin\theta_0^i - I_C\cos\theta_0^i\right) - I_D\cos^3\theta_0^i\right]
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
I_a &= \frac{1}{2}[b_1(\ln b_1 - 1) - a_1(\ln a_1 - 1) - b_2(\ln b_2 - 1) + a_2(\ln a_2 - 1)] \\
I_b &= b_0(1 - \ln b_0) - a_0(1 - \ln a_0) \\
I_A &= 0.5(I_a - I_p + I_q + I_r - I_s) \\
I_B &= \frac{a_0^3}{3}\left(\ln a_0 - \frac{1}{3}\right) - \frac{b_0^3}{3}\left(\ln b_0 - \frac{1}{3}\right) \\
I_C &= 0.5(I_p - I_q - I_r + I_s) \\
I_D &= I_b - I_B \\
I_p &= \frac{b_1^3}{3}\left(\ln b_1 - \frac{1}{3}\right) - b_1^2\left(\ln b_1 - \frac{1}{2}\right) + b_1(\ln b_1 - 1) \\
I_q &= \frac{a_1^3}{3}\left(\ln a_1 - \frac{1}{3}\right) - a_1^2\left(\ln a_1 - \frac{1}{2}\right) + a_1(\ln a_1 - 1) \\
I_r &= \frac{b_2^3}{3}\left(\ln b_2 - \frac{1}{3}\right) - b_2^2\left(\ln b_2 - \frac{1}{2}\right) + b_2(\ln b_2 - 1) \\
I_s &= \frac{a_2^3}{3}\left(\ln a_2 - \frac{1}{3}\right) - a_2^2\left(\ln a_2 - \frac{1}{2}\right) + a_2(\ln a_2 - 1) \\
a_0 &= \cos\left(\theta_1^i - \theta_0^i\right); \quad b_0 = \cos\left(\theta_2^i - \theta_0^i\right) \\
a_1 &= 1 + \sin\left(\theta_1^i - \theta_0^i\right); \quad b_1 = 1 + \sin\left(\theta_2^i - \theta_0^i\right) \\
a_2 &= 1 - \sin\left(\theta_1^i - \theta_0^i\right); \quad b_2 = 1 - \sin\left(\theta_2^i - \theta_0^i\right) \\
t_1 &= \frac{1}{3}\left(\sin^3\theta_1^i - \sin^3\theta_2^i\right); \quad t_2 = \frac{1}{3}\left(\cos^3\theta_1^i - \cos^3\theta_2^i\right)
\end{aligned} \tag{A.2}$$

References

[1] Y.-H. Chen, W.C. Chew, A three-dimensional finite difference code for the modeling of sonic logging tools, J. Acoust. Soc. Am. 93 (1998) 702–712.

[2] W.C. Chew, Q.H. Liu, Perfectly matched layers for elastodynamics: a new absorbing boundary condition, J. Comput. Acoust. 4 (1996) 341–359.

[3] C.J. Randall, Absorbing boundary condition for the elastic wave equation: velocity–stress formulation, Geophysics 54 (1989) 1141–1152.

[4] Q.-H. Liu, E. Schoen, F. Daube, C. Randall, H.-L. Liu, P. Lee, A three-dimensional finite difference simulation of sonic logging, J. Acoust. Soc. Am. 100 (1996) 72–79.

[5] I.R. Gonsalves, D.J. Shippy, F.J. Rizzo, Direct boundary integral equations for elastodynamics in three-dimensional half-spaces, Comput. Mech. 6 (1990) 279–292.

[6] F.J. Rizzo, D.J. Shippy, An advanced boundary integral equation method for three-dimensional thermoelasticity, Int. J. Numer. Method Eng. 11 (1977) 1753–1768.

[7] T.A. Cruse, An improved boundary-integral equation method for three-dimensional elastic stress analysis, Comput. Struct. 4 (1974) 741–754.

[8] F.J. Rizzo, D.J. Shippy, M. Rezayat, A boundary integral equation method for radiation and scattering of elastic waves in three dimensions, Int. J. Numer. Method Eng. 21 (1985) 115–129.

[9] Y. Liu, F.J. Rizzo, Hypersingular boundary integral equations for radiation and scattering of elastic waves in three dimensions, Comput. Method Appl. Method Eng. 107 (1993) 131–144.

[10] W.S. Hall, The Boundary Element Method, Kluwer Academic Publishers., London, 1994.

[11] G.D. Manolis, D.E. Beskos, Boundary Element Methods in Elastodynamics, Unwin Hyman, London, 1988.

[12] P.J. Schafbuch, R.B. Thompson, F.J. Rizzo, Application of the boundary element method to the elastic wave scattering by irregular defects, J. Nondestruct. Eval. 2 (1990) 113–127.

[13] P.S. Kondapalli, Frequency response of elastic bodies of revolution by the boundary element method, J. Acoust. Soc. Am. 98 (1995) 1558–1564.

[14] E.J. Nyström, Über die Praktische Auflösung von Linearen Integralgleichungen mit Anwendungen auf Randwertaufgaben der Potentialtheorie, Comment. Phys. Math. 4 (1928) 1–52.

[15] L.S. Canino, J.J. Ottusch, M.A. Stalzer, J.L. Visher, S. Wandzura, Numerical solution of the Helmholtz equation in 2D and 3D using a high-order Nyström discretization, Comput. Phys. 146 (1998) 627–663.
[16] S.D. Gedney, On deriving a locally corrected Nyström scheme from a quadrature sampled moment method, IEEE Trans. Antenna Propagat. 51 (2003) 2402–2412.
[17] S.D. Gedney, Application of the high-order Nyström scheme to the integral equation solution of electromagnetic interaction problems, 2000, IEEE Int. Symp. Electromag. Compat. 1 (2000) 289–294.
[18] M.G. Duffy, Quadrature over a pyramid or cube of integrands with a singularity at a vertex, SIAM J. Numer. Anal. 19 (1982) 1260–1262.
[19] Y.-H. Pao, V. Varatharajulu, Huygens' principle, radiation conditions, and integral formulas for the scattering of elastic waves, J. Acoust. Soc. Am. 59 (1976) 1361–1371.
[20] P.M. Morse, H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, New York, 1953.
[21] A.H. Stroud, Approximate Calculation of Multiple Integrals, Prentice-Hall, Englewood Cliffs, 1971.
[22] Y.-H. Pao, C.C. Mow, Scattering of plane compressional waves by a spherical obstacle, J. Appl. Phys. 34 (1963) 493–499.
[23] F.C. Moon, Y.-H. Pao, The influence of the curvature of spherical waves on dynamic stress concentration, ASME J. Appl. Mech. 34 (1967) 373–379.
[24] C.F. Ying, R. Truell, Scattering of a plane longitudinal wave by a spherical obstacle in an isotropically elastic solid, J. Appl. Phys. 27 (1956) 1086–1097.
[25] I.R. Gonsalves, D.J. Shippy, F.J. Rizzo, The direct boundary integral equation method for the three-dimensional elastodynamic transmission problem, Math. Comput. Model. 15 (1991) 155–164.
[26] P.S. Kondapalli, D.J. Shippy, The method of fundamental solutions for transmission and scattering of elastic waves, Comput. Method Appl. Method Eng. 96 (1992) 255–269.
[27] H.B. Dwight, Tables of Integrals and Other Mathematical Data, MacMillan, New York, 1961.